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Effect of spatial variation of thermal conductivity on non-fourier heat conduction in a finite slab †

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Abstract

The non-Fourier heat conduction problem in a finite slab is studied analytically. Dependence of thermal conductivity on space has been considered. The Laplace transform method is used to remove the time-dependent terms in the governing equation and the boundary conditions. The hyperbolic heat conduction (HHC) equation has been solved by employing trial solution method and collocation optimization criterion. Results show that the space-dependent thermal conductivity strongly affects the temperature distribution. A temperature peak on the insulated wall of the slab has been observed due to linear variation of thermal conductivity. It has been shown that the magnitude of the temperature peak increases with increasing the dimensionless relaxation time. To validate the approach, the results have been compared with the analytical solution obtained for a special case which shows a good agreement.

Keywords: Analytical; Hyperbolic; Heat conduction; Finite slab; Space-dependant thermal conductivity; Trial solution

1. Introduction

In highly unsteady situations and high heat flux applications such as laser pulse annealing of semiconductors, in order to avoid the failure of parabolic heat conduction equation based on the Fourier law, a new model of heat conduction was proposed by Cattaneo [1] and Vernotte [2]. In this model a finite speed of heat flux has been considered, which leads to the following equation:

$$\tau \frac{\partial q}{\partial t} + q = -k\nabla T \tag{1}$$

Where q is the heat flux vector and τ is the relaxation time. Introducing Eq. (1) into the energy equation leads to the hyperbolic heat conduction (HHC) equation.

$$\tau \rho c \frac{\partial^2 T}{\partial t^2} + \rho c \frac{\partial T}{\partial t} = \nabla.(k\nabla T) \tag{2}$$

Several analytical and numerical methods have been proposed to solve the one-dimensional and two-dimensional HHC equations in the finite slab with different boundary conditions. Ozisik [3] has done analytical solutions for the hyperbolic heat conduction equation describing the wave nature of thermal energy transport in a finite slab with insulated boundaries subjected to a volumetric energy source in the medium. The hyperbolic transient heat conduction equation in finite slab with insulated boundaries and arbitrary initial conditions was investigated analytically by Amin Moosaie [4]. Abdel-Hamid [5] solved the hyperbolic heat conduction equation under periodic surface disturbance using finite integral transform approach. Tang and Araki [6] studied the non-Fourier effect in a slab subjected to a periodic thermal disturbance by the method of Laplace transform.

Gembarovic [7] has researched analytically non-Fourier effects in a finite slab using the hyperbolic heat conduction model. In his work, the results for

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pulsed surface heat flux conditions were compared with those obtained from the standard parabolic heat conduction equation. Lewandowska [8] has presented an analytical solution of the hyperbolic heat conduction equation for the case of a thin slab symmetrically heated on both sides. Chen [9] has investigated two-dimensional hyperbolic heat conduction problems by using the hybrid numerical scheme. In this method, the time-dependent terms in the governing differential equations are removed by using the Laplace transform technique, and then the control volume method is used to discretize the space domain in the transform domain. There are also a number of works [10-13] which focus on the effect of temperature-dependent thermal conductivity on the various heat conduction problems.

In contrast to all previous works in which thermal conductivity has been assumed to be space-independent, in this paper the dependence of thermal conductivity on space which is mostly encountered in nonhomogeneous materials has been considered. Since this article is the first work towards the new assumption of space-dependent thermal conductivity, linear variation for thermal conductivity has been considered. It can be observed from the previous numerical works that the major difficulty encountered in the numerical solution of the HHC equation is numerical oscillation in the vicinity of the sharp discontinuities. When the thermal conductivity is a linear function of space, the problem becomes nonlinear. Thus, solving this problem numerically is a hard task. Finding an exact analytical solution for the nonlinear problems is not always possible, so that applying an approximate analytical solution to solve the problem will be useful. The trial solution method has been formulated to solve the general form of the HHC equation for the simple case of constant thermal conductivity by Kiwan [14], which has a good agreement with the exact solution. This method is also applicable for the case of variable thermal conductivity. In hyperbolic heat conduction, superposition of thermal waves results in some temperature peaks inside the slab. In contrast to homogeneous materials, these temperature peaks are intensified in nonhomogeneous materials due to variable thermal conductivity which affects its molecular structure.

The main objective of the present study is to find the temperature peaks and the time at which they occur. To apply the trial solution method, the partial differential equation is transferred from time domain to the Laplacian domain. In the next step, a polynomial distribution with unknown coefficients for the thermal field is considered which satisfies the boundary conditions. To find the polynomial coefficients, the collocation residual method as an optimization method is applied. In the final step, the temperature distribution is obtained using inverse Laplace transform.

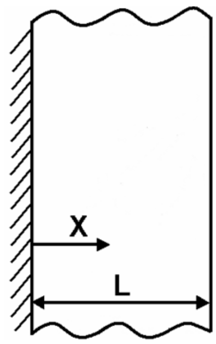

Fig. 1. Scheme of the studied geometry.

2. Problem description

To investigate the effect of variable thermal conductivity on hyperbolic heat conduction, an infinite one-dimensional plate with the thickness L has been considered, as shown in Fig. 1. The plate is initially at a uniform temperature T_∞ and suddenly the temperature of one side of the plate is raised to T_1 while the other side is maintained insulated.

In addition, this geometry can be considered as a plate with thickness of 2L which its two end sides experience the temperature rise to T_1 .

3. Formulation

The one-dimensional HHC equation is given as Eq. (3).

$$\begin{aligned}\tau\rho c\frac{\partial^2 T}{\partial t^2}+\rho c\frac{\partial T}{\partial t}&=\nabla.(k.\nabla T)\\ =\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x})&=\frac{\partial k}{\partial x}\cdot\frac{\partial T}{\partial x}+k\frac{\partial^2 T}{\partial x^2}\end{aligned} \tag{3}$$

The thermal conductivity of the slab is assumed to be a linear function of the direction X according to Eq. (4).

$$k=k_h(1+\lambda x) \tag{4}$$

Where k_h is the homogeneous thermal conductivity and λ is the parameter describing the variation of the thermal conductivity. Employing the following dimensionless parameters:

$$\theta = \frac{T - T_\infty}{T_1 - T_\infty}, \ \xi = \frac{x}{L}, \quad \eta = \frac{\alpha t}{L^2}, \quad \bar{\tau} = \frac{\alpha \tau}{L^2} \tag{5}$$

And supposing $\varepsilon = \lambda L$, thermal conductivity and Eq. (3) are rewritten as:

$$k = k_h(1 + \varepsilon\xi) \tag{6}$$

$$\bar{\tau}\frac{\partial^2\theta}{\partial\eta^2} + \frac{\partial\theta}{\partial\eta} = \frac{\varepsilon}{1+\varepsilon\xi}\cdot\frac{\partial\theta}{\partial\xi} + \frac{\partial^2\theta}{\partial\xi^2} \tag{7}$$

The initial and boundary conditions are written as:

$$\frac{\partial\theta}{\partial\xi}(0,\eta) = 0, \quad \theta(1,\eta) = 1$$

$$\frac{\partial\theta}{\partial\eta}(\xi,0) = 0, \quad \theta(\xi,0) = 0 \tag{8}$$

Taking the Laplace transform of Eq. (7) gives:

$$\bar{\tau} s^2\bar{\theta} + s\bar{\theta} = \frac{\varepsilon}{1+\varepsilon\xi}\cdot\frac{d\bar{\theta}}{d\xi} + \frac{d^2\bar{\theta}}{d\xi^2} \tag{9}$$

Where $\bar{\theta}$ is the Laplace transform of θ. The Laplace transforms of the boundary conditions are:

$$\bar{\theta}(1,s) = \frac{1}{s} \qquad \frac{d\bar{\theta}(0,s)}{d\xi} = 0 \tag{10}$$

4. Trial solution method

The trial solution method has been applied to solve the HHC equation based on polynomial shape function approximation. Second, third, and fourth-order approximations are used in this study. Only the results of the fourth-order polynomial shape function will be presented here. The proposed fourth-order approximation which satisfies the boundary conditions (10) is:

$$\tilde{\bar{\theta}} = \frac{1}{s} + a_1(\xi^2 - 1) + a_2(\xi^3 - 1) + a_3(\xi^4 - 1) + a_4(\xi^5 - 1) \tag{11}$$

The residual is obtained by substituting Eq. (11) into Eq. (9) which can be written as follows.

$$\begin{aligned}\text{Res} = {} & a_1[s\cdot(\tau s+1)(1+\varepsilon\xi)(\xi^2-1) - 2(1+\varepsilon\xi) - \\ & 2\varepsilon\xi] + a_2[s\cdot(\tau s+1)(1+\varepsilon\xi)(\xi^3-1) - 6\xi(1+\varepsilon\xi) - \\ & 3\varepsilon\xi^2] + a_3[s\cdot(\tau s+1)(1+\varepsilon\xi)(\xi^4-1) - \\ & 12\xi^2(1+\varepsilon\xi) - 4\varepsilon\xi^3] + a_4[s\cdot(\tau s+1). \\ & (1+\varepsilon\xi)(\xi^5-1) - 20\xi^3(1+\varepsilon\xi) - 5\varepsilon\xi^4] + \\ & (\tau s+1)(1+\varepsilon\xi)\end{aligned} \tag{12}$$

There are four unknown coefficients in Eq. (12). To obtain these coefficients, the collocation optimizing criterion has been applied. In this method, the residual must be set to zero at four equally spaced points $\xi = 0, \xi = 0.25, \xi = 0.5$ *and* $\xi = 0.75$ As a result, a system of algebraic equations will be obtained which can be easily solved. By inserting these coefficients into Eq. (11) and taking Laplace inversion, the temperature distribution will be obtained.

5. Results and discussion

Exact solution can be obtained by the method of separation of variables only for the simple case of constant thermal conductivity, $\varepsilon = 0$. A comparison between the exact analytical solution and the approximate solution at fixed dimensionless relaxation time for three different dimensionless times is shown in Fig. 2. It is clear from this figure that the predictions using fourth-order polynomial approximation have good agreement with the exact solution.

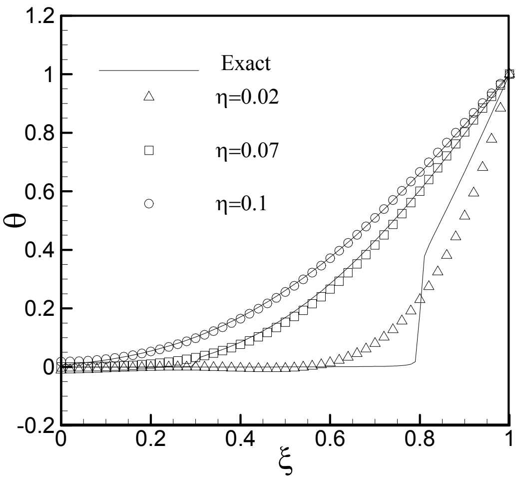

Fig. 2. Comparison between exact and approximate solutions at $\bar{\tau} = 0.01$ for different values of η.

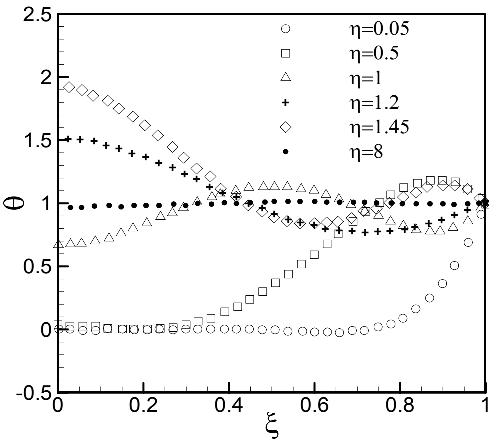

Fig. 3. Temperature distribution for $\bar{\tau}=1$ and $\varepsilon=1$ at different time intervals.

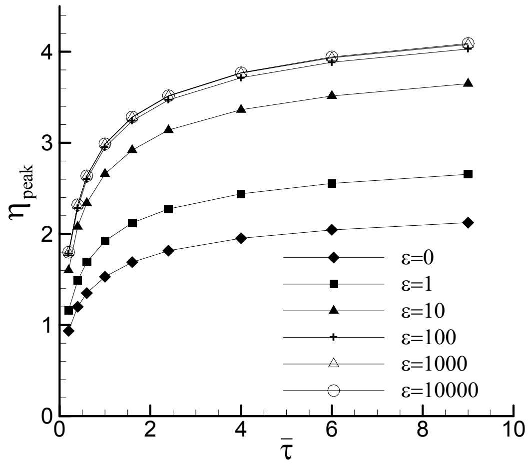

Fig. 4. Variation of temperature peak with $\bar{\tau}$ for different values of ε.

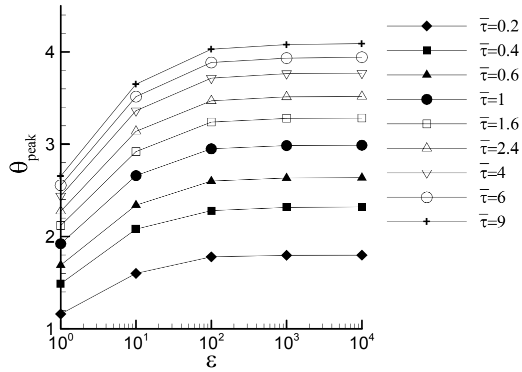

Fig. 5. Variation of temperature peak with ε for different values of $\bar{\tau}$.

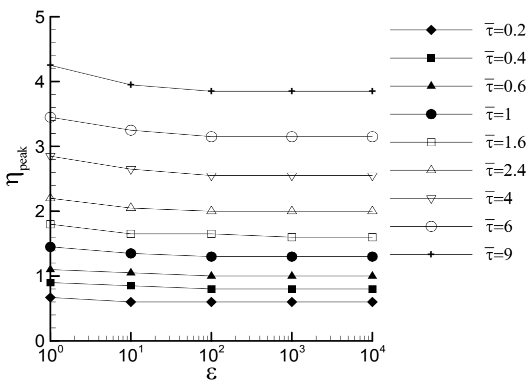

Fig. 6. Variation of dimensionless peak time with ε for different values of $\bar{\tau}$.

Temperature distribution in the slab for specific values of dimensionless relaxation time $\bar{\tau}=1$ and $\varepsilon=1$ at different time intervals has been shown in Fig. 3.

As shown in Fig. 3, at small values of time, only a small part of the slab has experienced the temperature change at the boundary. This part is in the vicinity of the wall with constant temperature of T_1. As the time increases, the dimensionless temperature at the insulated wall of the slab increases as well until it reaches a maximum amount greater than unity and then descends gradually to the amount of one, which is the steady state condition. This temperature peak is due to superposition of thermal waves and may affect the molecular structure of the plate.

In Fig. 4, variation of the temperature peak with respect to the dimensionless relaxation time for different values of ε has been depicted. It can be easily seen that for the amounts of ε larger than 100, increase of ε has almost no effect on the temperature peak.

The temperature peak also rises with increasing the dimensionless relaxation time as shown in Fig. 5.

It should be mentioned that for a constant value of ε, temperature peak has greater variations for small amounts of $\bar{\tau}$.

In Fig. 6 variation of the dimensionless time at which the temperature peak occurs with respect to ε for different values of dimensionless relaxation time has been shown. It is obvious from the figure that variation of ε has negligible effect on the temperature peak time and it mainly depends on the amount of dimensionless relaxation time.

6. Conclusion

Hyperbolic heat conduction in a finite slab with space-dependent thermal conductivity has been studied. Using Laplace transform, the HHC equation has been transferred to Laplacian domain. The obtained nonlinear differential equation has been solved using trial solution method based on second, third and fourth-order polynomial approximations. A temperature peak on the insulated wall of the slab has been observed due to the linear variation of thermal conductivity. The temperature peak increases with increasing the dimensionless relaxation time. For a constant value of dimensionless relaxation time, the temperature peak rises with increasing the slope of thermal conductivity function, but this slope has negligible effect on the temperature peak for amounts greater than a specific value. Also, variation of ε has negligible effect on the temperature peak time. It has been observed that temperature peak time mainly depends on the amount of dimensionless relaxation time.

Nomenclature

c : Specific heat $[J/kg.K]$
k : Thermal conductivity $[W/m.K]$
L : Length of the wall $[m]$
q : Heat flux per unit area $[W/m^2]$
s : Laplacian independent variable
t : Time $[s]$
T : Temperature $[K]$
x : The horizontal coordinate

Greek symbol

α : Thermal diffusivity $[m^2/s]$
θ : Dimensionless temperature $[\theta = \frac{T - T_\infty}{T_1 - T_\infty}]$
$\bar{\theta}$: Laplace transform of dimensionless temperature
$\tilde{\bar{\theta}}$: Polynomial approximation of $\bar{\theta}$
ε : Slope of thermal conductivity function multiplied by L
ξ : Dimensionless distance $[\xi = x/L]$
η : Dimensionless time $[\eta = \alpha t/L^2]$
τ : Relaxation time $[s]$
$\bar{\tau}$: Dimensionless relaxation time $[\bar{\tau} = \alpha\tau/L^2]$
λ : Slope of thermal conductivity function
ρ : Density $[kg/m^3]$

Subscripts

H : Homogeneous condition
1 : Condition at $x = L$
∞ : Condition at initial time

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